§5.2 Solving the honeycomb lattice model

The no-vortex sector

In Majorana form, the Hamiltonian can be written as:

$$H = H_1 + H_2$$

with
$$H_1 = \frac{i}{4} \sum_{ij} 2J\, u_{ij}\, c_i c_j ,$$
$$H_2 = \frac{i}{4} \sum_{ij} 2K \sum_{k} u_{ik} u_{jk}\, c_i c_j$$

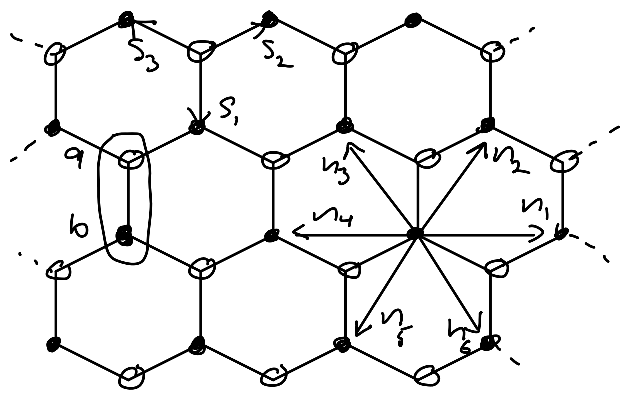
Focus on the case wher $u_{ij} = +1 \;\forall\, i,j$

$\rightarrow$ lowest energy state resides in this sector (Lieb's theorem)

Fourier transforming the Majoran modes, gives
$$a_{\vec r} = \sum_{\vec p} e^{-i\vec p\cdot\vec r} a_{\vec p}$$ (denote Majorana modes by a and b)

and similarly for $b_{\vec r}$ and $c_{\vec r}$

$$\rightarrow \quad H_1 = \frac{i}{4}\, 2J \sum_{\vec r} a_{\vec r}\left(b_{\vec r+\vec s_1} + b_{\vec r+\vec s_2} + b_{\vec r+\vec s_3}\right) + h.c.$$
$$= \frac{i}{4}\, 2J \sum_{\vec r, \vec p, \vec p'} \left(\sum_{\alpha=1,2,3} e^{-i\vec p\cdot\vec r - i\vec p'\cdot(\vec r+\vec s_\alpha)}\right) a_{\vec p}\, b_{\vec p'} + h.c.$$
$$= \frac{i}{4}\, 2J \sum_{\vec p} \left(\sum_{\alpha=1,2,3} e^{-i\vec p\cdot\vec s_\alpha}\right) a_{-\vec p}\, b_{\vec p} + h.c.$$



To simplify, define $\tilde{a} = e^{-i\pi/4} a$, $\tilde{b} = e^{i\pi/4} b$

and

$$f(\vec{p}) = 2J \sum_{\alpha=1,2,3} e^{-i\vec{p}\cdot\vec{s}_\alpha}$$

$$\longrightarrow \quad H_1 = \frac{1}{4} \sum_{\vec{p}} f(\vec{p})\, \tilde{a}^\dagger_{\vec{p}} \tilde{b}_{\vec{p}} + h.c.$$

Moreover, for next-to-nearest neighbour interactions we get

$$H_2 = \frac{iK}{2} \sum_{\vec{r},\vec{p},\vec{p}'} \Big(e^{-i\vec{p}'\cdot\vec{n}_1} - e^{-i\vec{p}'\cdot\vec{n}_2} + e^{i\vec{p}'\cdot(\vec{n}_1-\vec{n}_2)} - e^{i\vec{p}'\cdot\vec{n}_1} + e^{i\vec{p}'\cdot\vec{n}_2} - e^{-i\vec{p}'\cdot(\vec{n}_1-\vec{n}_2)} \Big)$$

$$\times e^{-i(\vec{p}+\vec{p}')\cdot\vec{r}}\, c_{\vec{p}}\, c_{\vec{p}'}$$

$$= \frac{1}{4} \sum_{\vec{p}} \Delta(\vec{p}) \left(\tilde{a}^\dagger_{\vec{p}} \tilde{a}_{\vec{p}} - \tilde{b}^\dagger_{\vec{p}} \tilde{b}_{\vec{p}} \right)$$

where

$$\Delta(\vec{p}) = 4K\left(-\sin\vec{p}\cdot\vec{n}_1 + \sin\vec{p}\cdot\vec{n}_2 + \sin\vec{p}\cdot(\vec{n}_1-\vec{n}_2) \right)$$

Combining H_1 and H_2, we get

$$H = \frac{1}{4} \sum_{\vec{p}} \left(\tilde{a}^{\dagger}_{\vec{p}} \ \tilde{b}^{\dagger}_{\vec{p}} \right) \underbrace{\begin{pmatrix} \Delta(\vec{p}) & f(\vec{p}) \\ f(\vec{p})^* & -\Delta(\vec{p}) \end{pmatrix}}_{=: H(\vec{p})} \begin{pmatrix} \tilde{a}_{\vec{p}} \\ \tilde{b}_{\vec{p}} \end{pmatrix}$$

$\rightarrow$ energy eigenvalue of Majorana fermions can be found by direct diagonalization of $H(\vec{p})$

For $K=0$ $(\Delta(\vec{p})=0)$, one gets

$$\mathcal{E}(\vec{p}) = \pm |f(\vec{p})| = \pm \frac{J}{2} \sqrt{1 + 4\cos^2 \frac{\sqrt{3}p_x}{2} + 4\cos \frac{3p_y}{2} \cos \frac{\sqrt{3}p_x}{2}}$$

"dispersion relation"

positive energy: conductance band
negative energy: valence band

$\mathcal{E}(\vec{p}) = 0$ for certain isolated values of momentum $\vec{p}$, known as Fermi pts.

example: $\vec{p}_{\pm} = \left(\frac{4\pi}{3\sqrt{3}}, 0 \right)$

Continuous limit approximation

The ground state is the state in which valence band is fully occupied

Continuum limit corresponds to large wave-length / small momentum limit

Set $K=0$:

expanding near the Fermi points, gives:

$$f(\vec{P}_+ + \vec{p}) = -3J(p_x + ip_y) + \mathcal{O}(\vec{p}^2)$$

$$f(\vec{P}_- + \vec{p}) = 3J(p_x - ip_y) + \mathcal{O}(\vec{p}^2)$$

$\longrightarrow$ substituting into the Hamiltonian:

$$H_+(\vec{p}) = H(\vec{P}_+ + \vec{p}) = 3J(-\sigma^x p_x + \sigma^y p_y)$$

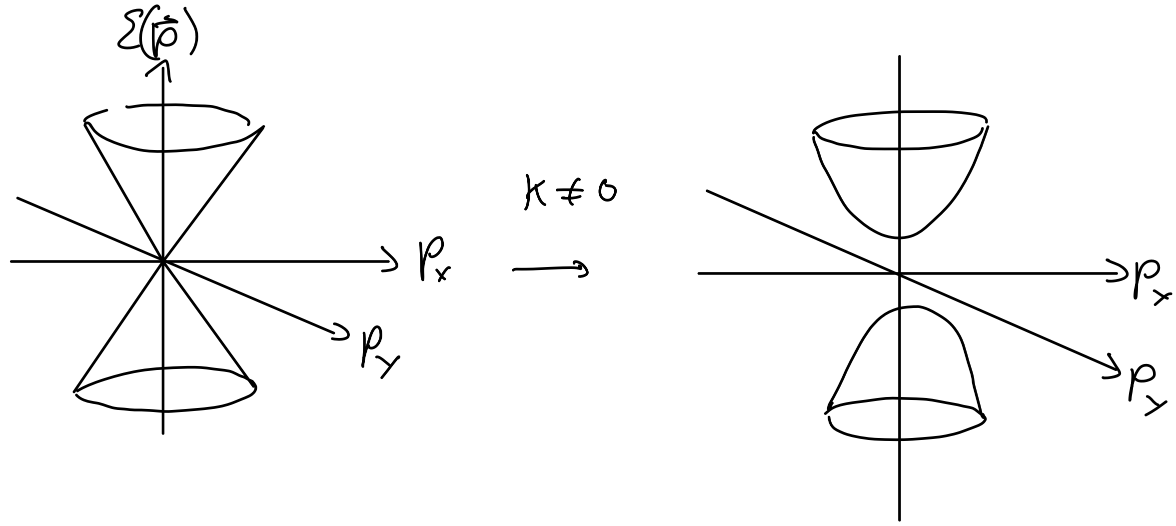
$$H_-(\vec{p}) = H(\vec{P}_- + \vec{p}) = 3J(\sigma^x p_x + \sigma^y p_y)$$

$K \neq 0$:

$$\Delta(\vec{P}_+ + \vec{p}) = 6\sqrt{3}K + \mathcal{O}(\vec{p}^2),$$

$$\Delta(\vec{P}_-) = -\Delta(\vec{P}_+)$$

$\longrightarrow$ creates energy gap $\Delta = 6\sqrt{3}K$ between valence and conductance band



Combining the two Hamiltonians $H_+(\vec{p})$ and $H_-(\vec{p})$ together and treating the two Fermi points as two pseudo-spin components gives in the basis $\Psi(\vec{p}) = (\tilde{a}_+, \tilde{b}_+, \tilde{b}_-, \tilde{a}_-)^T$:

$$H_{tot}(\vec{p}) = \begin{pmatrix} \Delta & -p & 0 & 0 \\ -\bar{p} & -\Delta & 0 & 0 \\ 0 & 0 & \Delta & p \\ 0 & 0 & \bar{p} & -\Delta \end{pmatrix}$$

$$= -\sigma^z \otimes \sigma^x p_x + \sigma^z \otimes \sigma^y p_y + \mathbb{1} \otimes \sigma^z \Delta$$

where $p = p_x + i p_y$, $\bar{p} = p_x - i p_y$

<u>note</u>: $\{\sigma^z \otimes \sigma^x, \sigma^z \otimes \sigma^y\} = 0$, $(\sigma^z \otimes \sigma^x)^2 = (\sigma^z \otimes \sigma^y)^2 = \mathbb{1}$

$\rightarrow$ Dirac operator with mass Δ

Topological properties:

Verify that the matrices

$$\Sigma_1 = \sigma^z \otimes \sigma^x, \quad \Sigma_2 = \sigma^z \otimes \sigma^y, \quad \Sigma_3 = \mathbb{1} \otimes \sigma^z$$

satisfy the su(2) algebra

$$[\Sigma_i, \Sigma_j] = 2i\,\varepsilon_{ijk}\,\Sigma_k$$

$$\rightarrow H_{tot}(\vec{p}) = \vec{\Sigma} \cdot \vec{n}(\vec{p}),$$

where $\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$

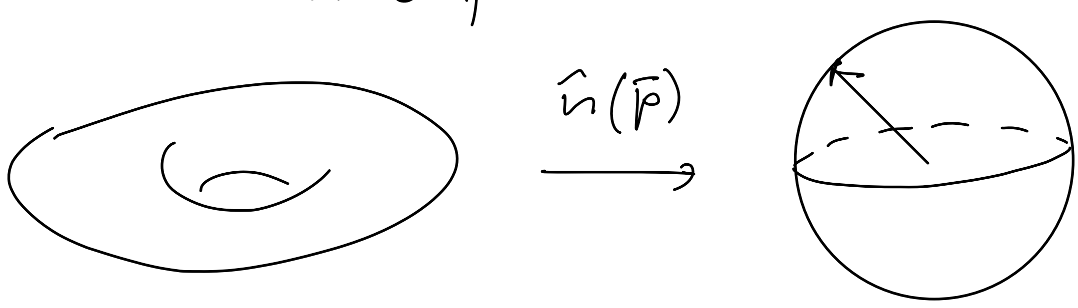
and $\vec{n}(\vec{p}) = (-p_x, p_y, \Delta)$

Consider the normalized vector $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$

Note that the momentum is periodic in both p_x and p_y $\rightarrow$ takes values on a torus

moreover: all possible values of $\hat{n}$ define the unit sphere

$\rightarrow$



One has :

- $\hat{n}(\vec{P}_+) = (0, 0, 1)$
- $\hat{n}(\vec{P}_-) = (0, 0, -1)$
- $\hat{n}(\vec{P} \mid |\vec{p}| \gg \Delta)$ is oriented along equator

$\longrightarrow$ for $\Delta \neq 0$, the map is one-to-one :

its Chern-number is

$$\nu = \frac{1}{4\pi} \iint dp_x \, dp_y \, \frac{\partial \hat{n}}{\partial p_x} \times \frac{\partial \hat{n}}{\partial p_y} \cdot \hat{n}$$

$$= \frac{\Delta}{|\Delta|} = \pm 1 \quad (\text{sign}(\Delta))$$

$\longrightarrow$ for $K \neq 0$ the Hamiltonian belongs to a topologically non-trivial class